Two approaches toward a high-efficiency flashing ratchet

V. M. Rozenbaum,[1,2,*] T. Ye. Korochkova,[2] D.-Y. Yang,[1] S. H. Lin,[1] and T. Y. Tsong[3,†]
[1]*Institute of Atomic and Molecular Sciences, Academia Sinica, P. O. Box 23-166, Taipei, Taiwan, Republic of China*
[2]*Institute of Surface Chemistry, National Academy of Sciences of Ukraine, Generala Naumova street 17, Kiev, 03164, Ukraine*
[3]*Institute of Physics, Academy of Sciences, Taipei, Taiwan, Republic of China*
and College of Biological Sciences, University of Minnesota, St. Paul, Minnesota 55108, USA
(Received 18 November 2004; published 7 April 2005)

For a flashing ratchet with periodic potentials fluctuating via random shifts of one-half period, a high efficiency is shown to result from two mechanisms. The previously reported one [Yu. A. Makhnovskii *et al.*, Phys. Rev. E **69**, 021102 (2004); V. M. Rozenbaum, JETP Lett. **79**, 388 (2004)] is realized in the near-equilibrium region and implies, first, the presence of a high barrier V_0 blocking the reverse movement of a Brownian particle and, second, identical, though energy-shifted, portions of the asymmetric flat potential profile on both half periods. We report another mechanism acting far from equilibrium, typical of strongly asymmetric potentials which are shaped identically on both half periods with a large energetic shift ΔV. The two mechanisms exhibit radically different limiting behavior of the maximum possible efficiency: $\eta_m \sim 1 - \exp(-\beta V_0/2)$ for the former and $\eta_m \sim 1 - \ln(2\beta\Delta V)/\beta\Delta V$ for the latter (β being the reciprocal temperature in energy units). The flux and the efficiency for a Brownian motor with a piecewise-linear potential are calculated using the transfer matrix method; an exact analytical solution can thus be obtained for an extremely asymmetric sawtooth potential, the simplest example of the second high-efficiency mechanism. As demonstrated, the mechanisms considered are also characteristic of a two-well periodic potential treated in terms of the kinetic approach.

DOI: 10.1103/PhysRevE.71.041102 PACS number(s): 05.40.−a, 05.60.Cd, 82.20.−w, 87.16.Nn

I. INTRODUCTION

Great recent interest in the mechanisms of high-efficiency energy conversion in living objects has given rise to a diversity of models, the favorite being the *flashing ratchet model*, or *flashing potential model*, of a Brownian motor [1–8]. This approach implies that a Brownian particle moves unidirectionally in an asymmetric periodic potential which is assumed to fluctuate by means of random (or regular) shifts by half a period L with the frequency γ [9–12]. Fluctuations of this kind can be caused by an external cyclic process generating the potential profile, such as an external signal or a far-from-equilibrium chemical reaction resulting in a conformational change of the particle or of the track [8,13–18]. It is just the fluctuations that are responsible for high efficiency of the motor, provided certain additional conditions are imposed on the potential shape. These are, first, the presence of a barrier blocking the backward motion of a Brownian particle on each period of the potential profile [11,12] and, second, the same potential shape, with a certain energy shift, on both half periods (this is a prerequisite for minimizing energy losses arising from the potential fluctuations) [12]. At first glance it would seem that the two conditions cannot be met at the same time. The following two instances, however, demonstrate their simultaneous realization, though with some reservations having no effect on the final result, the high efficiency.

The first example represents a two-well periodic potential with narrow wells, half a period apart, and separated by high barriers [11] [see Fig. 1(a)]. On switching between the potentials, a Brownian particle effects the transitions between the small vicinities of the potential well minima. The second example admits an arbitrarily shaped flat portion of the potential on the half period. At the same time, jump changes of the potential can occur at the half-period boundaries and one of the jumps can comprise, by virtue of a certain limiting transition, a high and narrow barrier [12] [see Fig. 1(b)]. In either instance, any switch (shift) of the potential results in movement of the particle to the right within a period $2L$ due to the establishment of quasiequilibrium on a $2L$-long portion of the asymmetric potential. It is additionally assumed that the average time between the potential switches is insufficiently long for the particle to surmount the high barriers separating the potential periods. Individual rightward displacements of the particle are summed up into a stationary flux directed to the right due to the potential shifts occurring at larger intervals than the average time needed for the establishment of quasiequilibrium in each potential. The stationary flux J is proportional to the frequency γ of the potential switches and generally approaches zero at $\gamma \to \infty$, i.e., when between the potential switches the particle cannot reach quasiequilibrium, which is necessary for the elementary acts of rightward displacements to occur. Importantly, directed motion of a Brownian particle persists even in the case that an external force F acts against the flux formed (though it is evident that at a certain value of F the flux J vanishes). The motor thus performs the useful work $W_{\text{out}} = 2LJF$ in a unit time, against the force F. The power W_{in} expended for this work is drawn from the potential-switching process and is proportional to the switching frequency γ. The motor efficiency is expressed as $\eta = W_{\text{out}}/W_{\text{in}}$.

*Email address: vrozen@mail.kar.net
†Email address: tsong@cbs.umn.edu

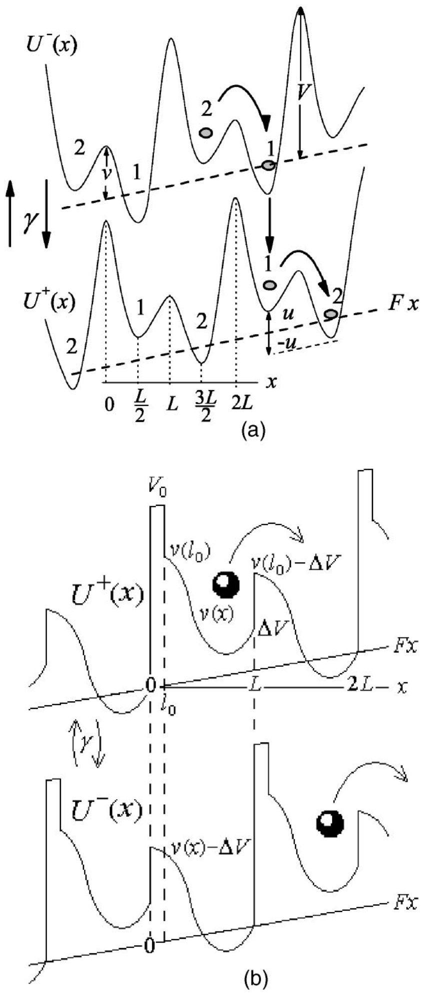

FIG. 1. The potentials $U^{\pm}(x)=V^{\pm}(x)+Fx$ that are contributed by the asymmetric periodic components $V^{\pm}(x)=V^{\pm}(x+2L)=V^{\mp}(x+L)$ shifting by half a period L with frequency γ and by the external field of the load force F. (a) A two-well potential profile with the wells spaced at a half period and separated by high barriers [11]. (b) Each potential comprises a high barrier V_0 in the narrow region l_0. Both half periods L contain the same flat portion $v(x)$ with the energy shift ΔV. Periodicity of the function $V^{\pm}(x)$ is realized via its jumps. (The potential curves and boundary points on them are indicated with reference to the straight line Fx [12].)

Until recently most attention was concentrated on the sawtooth potential, which is simple enough for modeling purposes and characterized by a minimum number of varied parameters, viz., the amplitude and the asymmetry parameter. Due to its linear shape, the differential equations accounting for the diffusion and the drift of a Brownian particle admit analytical solutions because their coefficients are constants. The previous treatment concerned the fluctuations of such a potential between two states differing only in its amplitude [1]. If the potential is absent completely in one of the states, such a state implies pure diffusion motion of a particle; as a result, the maximum efficiency value, $\eta_m \sim 0.05$, is rather small [6,19]. If the potential switches between two states, with their spatial periods equal and their extrema shifted relative to each other to a certain degree, Brownian motion provides no contribution at all to the generation of the directed movement of the particle [3]. Hence an avenue to increased motor efficiency arises. The potential switches are conveniently treated as repeated shifts of the same potential by half a period. With a two-well potential thus considered, the motor has the best operating characteristics [18]. It was stated in Ref. [10] that high efficiency in a sawtooth potential is reached provided a Brownian particle switches only between the restricted areas of two half-period-shifted potential counterparts, namely, between the narrow vicinities of the potential minima. The question of whether a high-efficiency motor is possible outside of this regime remained open. Further, it was proved that such shifts (switches) of the extremely asymmetric sawtooth potential give rise to a rather small maximum value η_m of the function $\eta(F)$ both for low frequencies γ (when $\eta_m \propto \gamma$) and for high frequencies (when $\eta_m \propto \gamma^{-1/2}$) [12]. It was still unclear whether the maximum of the function $\eta_m(\gamma)$ can tend to unity.

One of the main inferences of the present work is that the maximum efficiency can approach unity by the expression $\eta_m \to 1-\ln(2\beta\Delta V)/\beta\Delta V$ at $\beta\Delta V \gg 1$, where $2\Delta V$ is the amplitude of the extremely asymmetric sawtooth potential and $\beta=(k_B T)^{-1}$ is the reciprocal temperature (k_B is the Boltzmann constant and T is the absolute temperature). This result was obtained using the transfer matrix method (Sec. II) which enables an arbitrary potential profile to be considered as a set of small portions described by linear functions. The main characteristics of the motor operating at the expense of the sawtooth potential shifting fluctuations are calculated as functions of many parameters, such as the load force F, the shifting frequency γ, the amplitude $2\Delta V$, and the potential asymmetry l (Sec. III). The best motor is the one with an extremely asymmetric sawtooth potential; its flux and efficiency can be derived analytically. The limiting behavior found for the efficiency shows drastic distinctions from that in the case when an additional barrier is involved which blocks the backward motion of the Brownian particle. Consideration of this effect leads us away from the treatment of specifically shaped potentials to a more general problem, design of high-efficiency Brownian motors using certain structural elements of the potential profile (Sec. IV).

II. TRANSFER MATRIX METHOD

The dynamics of Brownian particle motion in the potentials $U^{\pm}(x)=V^{\pm}(x)+Fx$ [the superscripts "+" and "−" denote the potentials with half-period-shifted components $V(x)$ and the relevant quantities] are determined by two distribution functions $\rho^{\pm}(x,t)$ which obey the Smoluchowski equation [20] with an additional term accounting for the random transitions of a particle between the potentials $U^{\pm}$ with the frequency γ:

$$\frac{\partial \rho^{\pm}(x,t)}{\partial t} = -\frac{\partial j^{\pm}(x,t)}{\partial x} - \gamma[\rho^{\pm}(x,t) - \rho^{\mp}(x,t)]. \qquad (1)$$

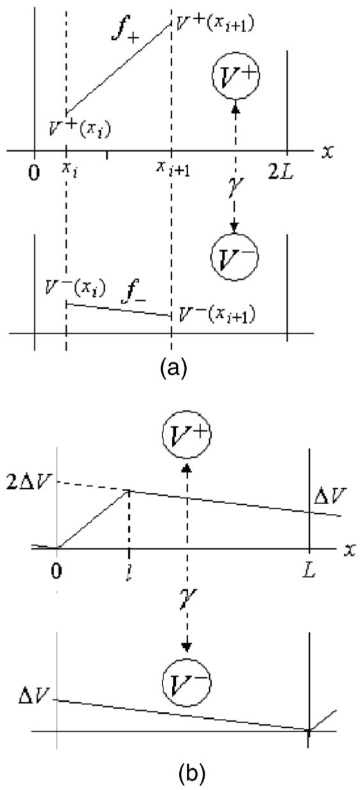

FIG. 2. (a) Linear potentials mutually switching with the frequency γ on the interval (x_i, x_{i+1}). (b) Switching sawtooth potentials on a half-period interval.

The fluxes $j^{\pm}(x,t)$ are given here by the expression

$$j^{\pm}(x,t) = -De^{-\beta U^{\pm}(x)}\frac{\partial}{\partial x}[e^{\beta U^{\pm}(x)}\rho^{\pm}(x,t)] \tag{2}$$

where D is the diffusion coefficient. In the stationary state, the total flux $J \equiv j^+(x) + j^-(x)$ is constant. It specifies the useful work $W_{\text{out}} = 2FLJ$ done by the motor against a load force F in unit time. The energy expended for switching the potentials $U^+ \to U^- \to U^+$ in unit time can be written as follows [11,12,19]:

$$W_{\text{in}} = \gamma \int_0^{2L} [V^+(x) - V^-(x)][\rho^-(x) - \rho^+(x)]dx. \tag{3}$$

These relations govern the motor efficiency $\eta = W_{\text{out}}/W_{\text{in}}$. We point out that there are a variety of alternative ways to define the input energy (and, hence, the efficiency) depending on the problem statement (see, e.g., Refs. [7,10] and the literature therein). In what follows, we apply the definition (3) because it has a natural relation to the model considered and requires no additional assumptions about the source of the input energy.

The main characteristics of a Brownian motor are conveniently calculated using so-called transfer matrices $\hat{T}(x_{i+1} - x_i)$ describing the transitions between the state vectors $\hat{\xi}(x_i)$ and $\hat{\xi}(x_{i+1})$,

$$\hat{\xi}(x_{i+1}) = \hat{T}(x_{i+1} - x_i)\hat{\xi}(x_i), \tag{4}$$

which are defined as

$$\hat{\xi}(x) = \begin{pmatrix} \rho^+(x) - \rho^-(x) \\ \rho^+(x) + \rho^-(x) \\ -[j^+(x) - j^-(x)]/D \\ -J/D \end{pmatrix}. \tag{5}$$

If the period of functions $V^{\pm}(x)$ is partitioned into sufficiently small portions, the potentials $U^{\pm}(x) = V^{\pm}(x) + Fx$ can be regarded as linear functions within each portion (x_i, x_{i+1}) [see Fig. 2(a)]:

$$V^{\pm}(x) + Fx = V^{\pm}(x_i) + f_{\pm,i}(x - x_i),$$

$$f_{\pm,i} \equiv F + [V^{\pm}(x_{i+1}) - V^{\pm}(x_i)]/(x_{i+1} - x_i). \tag{6}$$

Then the system of equations (1), (2) is equivalent to a differential equation of first order in the vector state $\hat{\xi}(x)$:

$$\frac{d}{dx}\hat{\xi}(x) = \hat{\epsilon}\hat{\xi}(x), \quad \hat{\epsilon} = \begin{pmatrix} -(f_+ + f_-)/2 & -(f_+ - f_-)/2 & 1 & 0 \\ -(f_+ - f_-)/2 & -(f_+ + f_-)/2 & 0 & 1 \\ 2\tilde{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

where $\tilde{\gamma} = \gamma/D$. [From here on, we assume, for conciseness, $\beta = (k_BT)^{-1} = 1$ and omit the subscript i from the symbol $f_{\pm}$, wherever this shorthand notation is unambiguous.] The solution for $\hat{\xi}(x)$ can be written in the following form:

$$\hat{\xi}(x) = \hat{C}\begin{pmatrix} A_1\exp(\lambda_1 x) \\ A_2\exp(\lambda_2 x) \\ A_3\exp(\lambda_3 x) \\ A_4\exp(\lambda_4 x) \end{pmatrix}, \tag{8}$$

with the matrix $\hat{C}$ specified as

$$\hat{C} = \begin{pmatrix} 0 & \tilde{\gamma}^{-1}(f_+ + \lambda_2)\lambda_2 & \tilde{\gamma}^{-1}(f_+ + \lambda_3)\lambda_3 & \tilde{\gamma}^{-1}(f_+ + \lambda_4)\lambda_4 \\ 2 & 2 - \tilde{\gamma}^{-1}(f_+ + \lambda_2)\lambda_2 & 2 - \tilde{\gamma}^{-1}(f_+ + \lambda_3)\lambda_3 & 2 - \tilde{\gamma}^{-1}(f_+ + \lambda_4)\lambda_4 \\ f_+ - f_- & 2(f_+ + \lambda_2) & 2(f_+ + \lambda_3) & 2(f_+ + \lambda_4) \\ f_+ + f_- & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

and A_i ($i=1, 2, 3, 4$) denoting arbitrary coefficients; λ_i are the eigenvalues of the matrix $\hat{\varepsilon}$, with $\lambda_1=0$ and the other ones satisfying the equation

$$\lambda^3 + (f_+ + f_-)\lambda^2 + (f_+ f_- - 2\tilde{\gamma})\lambda - \tilde{\gamma}(f_+ + f_-) = 0. \quad (10)$$

To calculate the transfer matrix, the coefficients A_i are conveniently expressed in terms of the vector $\hat{\xi}(0)$:

$$A_i = \sum_{j=1}^{4} (\hat{C}^{-1})_{ij}\xi_j(0). \quad (11)$$

Then we use the transfer matrix $\hat{T}(x)$ to go from the vector $\hat{\xi}(0)$ to $\hat{\xi}(x)$:

$$\hat{\xi}(x) = \hat{T}(x)\hat{\xi}(0), \quad \hat{T}(x) = \hat{C}\hat{E}(x)\hat{C}^{-1}, \quad E_{ij}(x) = \exp(\lambda_j x)\delta_{ij}. \quad (12)$$

If the period $2L$ of the functions $V^{\pm}(x)$ contains N intervals (x_i, x_{i+1}), then, by virtue of the boundary condition $\hat{\xi}(2L) = \hat{\xi}(0)$, we arrive at

$$[\hat{1} - \hat{T}(2L - x_{N-1}) \cdots \hat{T}(x_{i+1} - x_i) \cdots \hat{T}(x_1)]\hat{\xi}(0) = 0. \quad (13)$$

The bracketed matrix has a determinant equal to zero due to the fact that the full flux J (the component ξ_4) is the same on each interval (x_i, x_{i+1}). Hence the system of equations (13) has a nontrivial solution $\hat{\xi}(0)$ which is defined to an arbitrary constant calculable from the normalization condition

$$\int_0^{2L} [\rho_+(x) + \rho_-(x)]dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_i+1} \xi_2(x)dx = 1. \quad (14)$$

Here we set $x_0=0$ and $x_N=2L$, and also assume that $\xi_2(x)$ on the interval (x_i, x_{i+1}) is related to $\hat{\xi}(0)$ as follows:

$$\hat{\xi}(x) = \hat{T}(x - x_i)\hat{T}(x_i - x_{i-1}) \cdots \hat{T}(x_1)\hat{\xi}(0). \quad (15)$$

The matrices $\hat{T}(x-x_i)$ enable straightforward integration on account of their simple exponential x dependence [see Eq. (12)]. The solution $\hat{\xi}(0)$ gives the desired flux $J=-D\xi_4(0)$ and allows, with regard to Eqs. (3), (6), and (15), calculation of the energy expended:

$$W_{\text{in}} = -\frac{D}{2}\sum_{i=0}^{N-1} [V^+(x_i) - V^-(x_i)][\xi_3(x_{i+1}) - \xi_3(x_i)] - \frac{D}{2}\sum_{i=0}^{N-1} (f_{+,i} - f_{-,i})\left[(x_{i+1} - x_i)\xi_3(x_{i+1}) - \int_{x_i}^{x_{i+1}} \xi_3(x)dx\right]. \quad (16)$$

It is notable that the developed calculation strategy is significantly simplified at $f_+=f_-\equiv f$. Then the roots of the cubic equation (10) can be written in a concise analytical form:

$$\lambda_2 = -f, \quad \lambda_{3,4} = -\frac{1}{2}(f \pm \Delta), \quad \Delta \equiv \sqrt{f^2 + 8\tilde{\gamma}}, \quad (17)$$

and the nonzero matrix elements $\hat{T}(x)$ (12) become

$$T_{11,33}(x) = [\mp f \sinh(x\Delta/2)/\Delta + \cosh(x\Delta/2)]\exp(-fx/2),$$

$$T_{13}(x) = [2 \sinh(x\Delta/2)/\Delta]\exp(-fx/2),$$

$$T_{31}(x) = [4\tilde{\gamma} \sinh(x\Delta/2)/\Delta]\exp(-fx/2),$$

$$T_{22}(x) = \exp(-fx), \quad T_{24}(x) = [1 - \exp(-fx)]/f, \quad T_{44}(x) = 1. \quad (18)$$

With expression (16), the contributions to W_{in} from the intervals (x_i, x_{i+l}) are proportional to the energy shift $V^+(x_i) - V^-(x_i)$ if $f_{+,i}=f_{-,i}$.

III. SAWTOOTH POTENTIALS SHIFTED BY HALF A PERIOD

Consider the sawtooth potentials $V^{\pm}(x)$ shown in Fig. 2(b). By virtue of the equality $V^{\pm}(x+L)=V^{\mp}(x)$, the consideration can be restricted to the region $(0,L)$ which is partitioned into two portions $(0,l)$ and (l,L). On each interval, the potentials are written as linear functions (6), the boundary condition (13) within a half period assuming the form

$$[\hat{R} - \hat{T}(L - l)\hat{T}(l)]\hat{\xi}(0) = 0, \quad \hat{R} = \begin{pmatrix} \hat{\sigma} & \hat{0} \\ \hat{0} & \hat{\sigma} \end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (19)$$

[Here we invoke the identity $\hat{\xi}(L)=\hat{R}\hat{\xi}(0)$.] On the interval (l,L), the following conditions are valid:

$$f_+ = f_- \equiv f = F - \Delta V/L, \quad V^-(x) = \Delta V(L - x)/L,$$
$$V^+(x) = V^-(x) + \Delta V, \quad (20)$$

and the corresponding transfer matrix is representable analytically [see Eq. (18)]. On the interval $(0,l)$, the quantities f_- and $V^-(x)$ are given by the same relations (20), whereas

$$f_+ = F + \Delta V(2L - l)/lL, \quad V^+(x) = [\Delta V(2L - l)/lL]x, \quad (21)$$

and the corresponding transfer matrix is only numerically calculable.

The main motor characteristics are expressible analytically for an extremely asymmetric sawtooth potential at $l \to 0$. In this case, taking into account the equalities $j^{\pm}(l) = j^{\pm}(0) = j^{\mp}(L)$ and $\rho^{\pm}(0) = \rho^{\mp}(L)$, and also the boundary conditions for the jump $\rho^+(x)$ and continuity $\rho^-(x)$ at the point $x=0$,

$$\rho^+(0) = e^{2\Delta V}\rho^+(l), \quad \rho^-(0) = \rho^-(l), \quad l \to 0, \quad (22)$$

the transfer matrix on the interval $(0,l)$ is written as

$$\hat{T}(l) = \begin{pmatrix} \hat{t} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}, \quad \hat{t} = \frac{1}{2}\begin{pmatrix} e^{-2\Delta V}+1 & e^{-2\Delta V}-1 \\ e^{-2\Delta V}-1 & e^{-2\Delta V}+1 \end{pmatrix}. \quad (23)$$

Thus, the transfer matrices on both intervals are obtained in the analytical representation, which enables us, with some cumbersome transformations, to arrive at

$$J = \frac{DA}{2(Z_1 B - Z_2 A)}, \quad \eta = \frac{FL}{\Delta V}\frac{f}{1-e^{-fL}}\frac{A}{2\tilde{\gamma}\sinh \Delta V}, \quad (24)$$

where

$$A = \Psi_0 e^{\Delta V}(e^{-FL}\cosh \Delta V - 1) + \Psi_L(e^{-FL} - \cosh \Delta V),$$

$$B = e^{-FL}(\Psi_0 e^{\Delta V}\cosh \Delta V + \Psi_L),$$

$$Z_1 = \frac{4}{f^2}\sinh^2(fL/2), \quad Z_2 = \frac{1}{f^2}(fL + e^{-fL} - 1),$$

$$\Psi_{0,L} = \frac{1}{2\sinh L\Delta/2}\{[\exp(\pm fL/2) + \cosh(L\Delta/2)]\Delta \pm f \sinh L\Delta/2\}. \quad (25)$$

Figure 3 demonstrates the dependences of the flux J and the efficiency η on the load force F; they were calculated by relationships (24) and (25) with different values of the parameters $\tilde{\gamma}L^2$ and $\beta\Delta V$. As seen, the Brownian motor under consideration, like a motor of any kind, exhibits a monotonic decrease in J with rising F and a nonmonotonic behavior of $\eta(F)$. For an extremely asymmetric sawtooth potential at $F=0$, the flux J monotonically increases with the frequency γ and reaches saturation (a certain nonzero value) at $\gamma \to \infty$ [see the solid line in Fig. 4(a)]. The asymptotic behavior of this dependence is described by the following expressions:

$$J \underset{\gamma\to 0}{\to} \frac{\gamma}{2}\frac{\sinh \Delta V - \Delta V}{1+\cosh \Delta V},$$

$$J \underset{\gamma\to\infty}{\to} \frac{D}{2L^2}\frac{\Delta V^2 \sinh(\Delta V/2)}{\Delta V \sinh(\Delta V/2) + 2\cosh(\Delta V/2)}. \quad (26)$$

For both asymptotics, the flux is proportional to $(\beta\Delta V)^3$ at $\beta\Delta V \ll 1$. With $\beta\Delta V \gg 1$ and low frequencies γ of the potential switching, we obtain $J \to \gamma/2$, that is, the Brownian particle travels, on the average, the distance L in the time γ^{-1} (the average velocity is given by the product $2LJ$). Likewise, the physical meaning is apparent for the relationship $2LJ \to (\beta D)(\Delta V/L)$ at $\beta\Delta V \gg 1$ and large values of γ: the average velocity is equal to the ratio of the force $(\Delta V/L)$ to the friction coefficient $(\beta D)^{-1}$.

A nonzero asymptotic of the flux at $\gamma \to \infty$ arises from a jump change of the potential [12]. For a sawtooth potential with a nonzero value of the parameter l, a well-known general regularity holds which implies that $J \to 0$ at $\gamma \to \infty$ [7] [see also the dependences indicated by the dashed lines in Fig. 4(a) which were calculated numerically using the above-described transfer matrix formalism]. Thus, the γ dependence of J becomes nonmonotonic; the flux is weaker and it vanishes faster with rising frequency the larger are the values of l. The maximum of the function $J(\gamma)$ also decreases with an increase in l (see the solid line in Fig. 5).

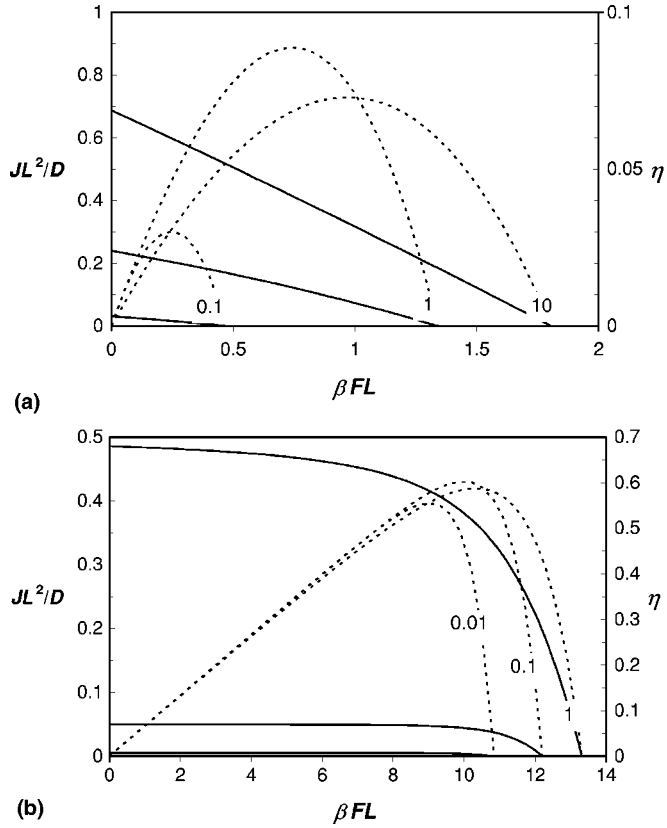

FIG. 3. The flux J (solid lines, the left axis) and the efficiency η (dashed lines, the right axis) versus the load force F for an extremely asymmetric sawtooth potential at varied values of $\tilde{\gamma}L^2$ (indicated on the curves) and constant values of the parameter $\beta\Delta V = 3$ (a) and 15 (b). The dependences are calculated by Eqs. (24) and (25).

It is noteworthy that the frequency dependence of the efficiency is nonmonotonic (see Fig. 3), which was pointed out previously [12] for a variety of potential profiles. The γ dependences of the function $\eta(F)$ maxima are shown in Fig. 4(b). The largest values of the efficiency are attained for an extremely asymmetric sawtooth potential (see the solid line). The function $\eta_m(\gamma)$ is proportional to γ at $\gamma \to 0$ and to $\gamma^{-1/2}$ at $\gamma \to \infty$ [12]. For instance, at $\beta\Delta V \ll 1$ the corresponding asymptotics takes the form

$$\eta_m(\gamma) \approx \begin{cases} (\beta\Delta V)^4 \tilde{\gamma}L^2/288, & \gamma \to 0, \\ (\beta\Delta V)^4/(32\sqrt{2\tilde{\gamma}}L), & \gamma \to \infty. \end{cases} \quad (27)$$

The efficiency is reduced with decreasing asymmetry, i.e., with rising parameter l (see the dashed lines). As a function of two variables F and γ, the efficiency η has its maximum values dependent only upon the asymmetry parameter l/L and the parameter $\beta\Delta V$ accounting for the ratio of the potential amplitude to temperature. The corresponding dependences for potentials of varied asymmetry appear as in Figs. 5 and 6. Thus, it is an extremely asymmetric potential

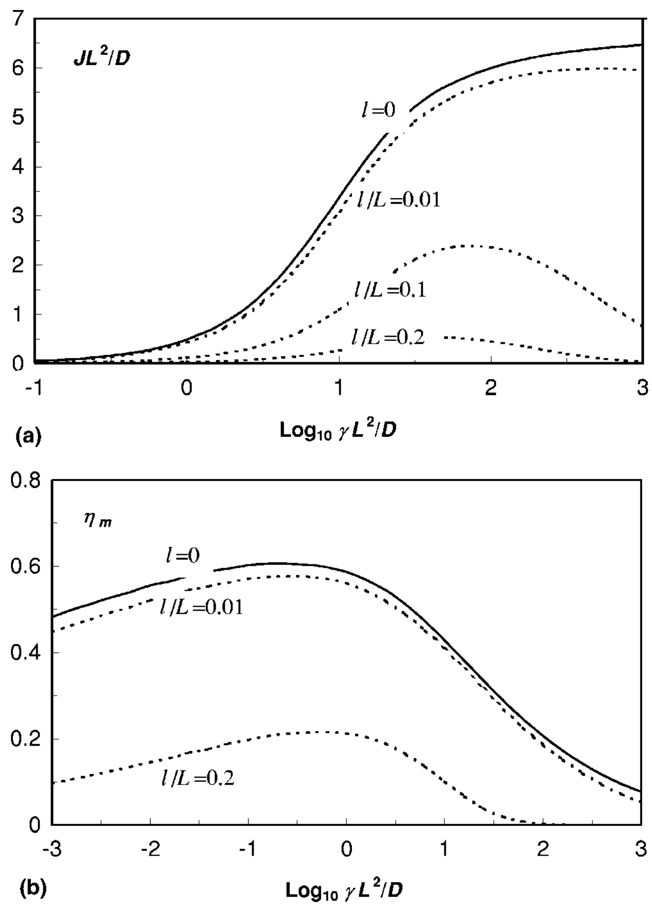

FIG. 4. The flux J at $F=0$ (a) and the maximum η_m of the function $\eta(F)$ (b) versus the potential switching frequency γ for potentials of varied asymmetry at $\beta\Delta V=15$. The dependences for the extremely asymmetric sawtooth potential at $l=0$ are calculated by Eqs. (24) and (25) and represented by the solid lines. The same dependences for the potential with the varied asymmetry parameter l/L are calculated numerically using the transfer matrix method described in Sec. II and represented by the dashed lines, with l/L values indicated near them.

that provides the best characteristics among the sawtooth potentials.

As the parameter ΔV increases (or the temperature T is lowered), the blocking effect of the barrier on the backward flux is enhanced and the efficiency grows as shown in Fig. 6. It is therefore appropriate to analyze the maximum possible values of the efficiency starting from the limiting behavior of relations (24) and (25) at $\beta\Delta V \gg 1$. Assume that the following inequalities can be met simultaneously:

$$\Delta V \gg \varphi \equiv \Delta V - FL \gg 1, \quad \Gamma \equiv \tilde{\gamma}L^2 \ll \varphi. \qquad (28)$$

Then we have

$$\eta(\varphi,\Gamma) \approx (1-\varphi/\Delta V)[1-2\varphi\exp(-\varphi) - 2\Gamma/\varphi^2 - 2\varphi^2 \times\exp(-2\varphi)/\Gamma]. \qquad (29)$$

The maximum η_m of the function of two variables $\eta(\varphi,\Gamma)$, and the condition for its occurrence are written as follows:

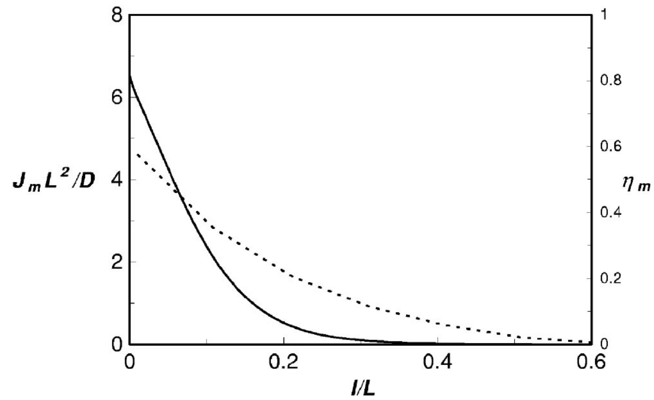

FIG. 5. The maximum J_m of the function $J(F=0,\gamma)$ (solid lines, the left axis) and the maximum η_m of the function $\eta(F,\gamma)$ (dashed lines, the right axis) versus the asymmetry parameter l/L at $\beta\Delta V=15$. The dependences are calculated numerically by the transfer matrix method (see Sec. II).

$$\eta_m = \frac{[1-2(\varphi_m+1)\exp(-\varphi_m)]^2}{1+2(\varphi_m^2-2)\exp(-\varphi_m)},$$
$$\frac{(\varphi_m+1)\exp(-\varphi_m)}{1+2(\varphi_m^2-2)\exp(-\varphi_m)} = \frac{1}{2\Delta V}, \quad \Gamma_m = \varphi_m^2\exp(-\varphi_m). \qquad (30)$$

In the limit $\Delta V \to \infty$, we arrive at

$$\eta_m \to 1 - \frac{\ln 2\Delta V}{\Delta V}, \quad \varphi_m \to \ln 2\Delta V,$$
$$\Gamma_m \to \frac{\ln 2\Delta V}{2\Delta V}, \quad J(\varphi_m,\Gamma_m) \to \frac{D}{2L^2}\Gamma_m. \qquad (31)$$

It is readily seen that all inequalities (28) are satisfied by relations (31), so that the starting prerequisites to the derivation of the asymptotic (29) prove to be adequate. Note that with the efficiency tending to unity, the corresponding flux (and hence the useful work W_{out}) tends to zero. On the other hand, a large flux is realized at a low efficiency. These trends are typical of any motor: the maximum values of W_{out} and η_m never occur simultaneously.

IV. DISCUSSION AND CONCLUSIONS

Two conditions were formulated previously [12] for a high efficiency of the flashing ratchet with periodic potentials fluctuating by means of random shifts by half a period: "First, a high narrow barrier V_0 exists that blocks the reverse flux. Second, a smooth potential profile of an arbitrary shape is repeated with the energy shift ΔV on both half periods of the function $V^{+}(x)$." With these conditions met, the maximum possible efficiency is estimated as $\eta_m \sim 1 - \exp(-\beta V_0/2)$. Here we analyze their applicability to the model developed in the present work.

The equally inclined ($f_+ = f_-$) piecewise-linear portions of the switching potentials $U^{\pm}$ can be regarded as identically shaped ones. They can be shifted relative to each other only by a fixed energy value. It is for such portions of the poten-

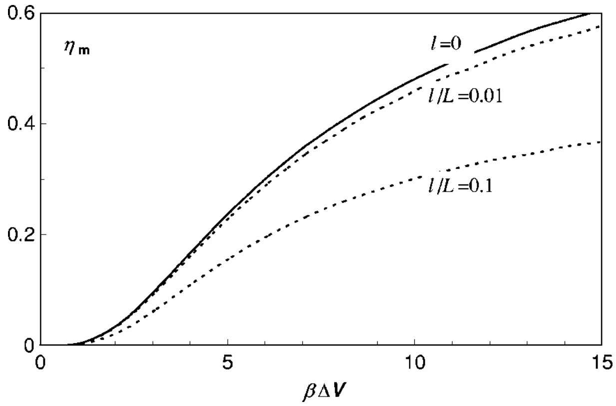

FIG. 6. The maximum η_m of the function $\eta(F,\gamma)$ versus the parameter $\beta\Delta V$ for potentials of varied asymmetry. The extremely asymmetric sawtooth potential at $l=0$ is represented by the solid lines. The values of the asymmetry parameter l/L are indicated near the dashed lines. The dependences are calculated and depicted as in Fig. 4.

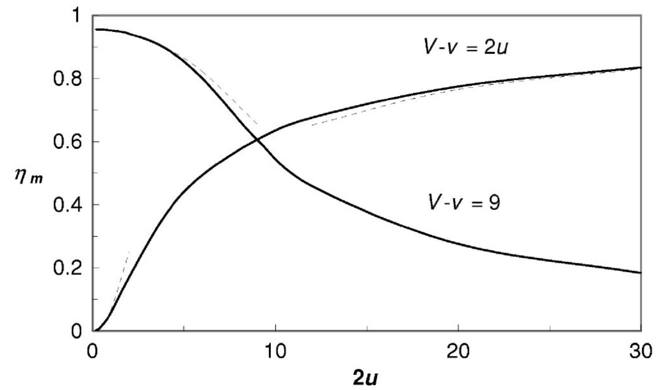

FIG. 7. The maximum efficiency η_m versus the well depth difference $2u$ of a two-well potential for a constant barrier height difference $V-v=9$, and in the case when the potential profile is shaped identically on both half periods with the energy shift $V-v=2u$. The dependences exactly calculated by Eq. (32) are shown by the solid lines. The asymptotics mentioned in the text and calculated by Eqs. (33) and (34) are represented by the dashed lines.

tial that transfer matrices (18) are representable analytically. In the case of the sawtooth potential, a portion of this kind is represented by the interval (l,L). At $l\to 0$, the transfer matrix of the interval $(0,l)$ also has the analytical representation (23), which enables the exact analytical solution (24) to be obtained. Thus, the possibility to analytically describe an extremely asymmetric sawtooth potential is attributed to the fact that it is shaped identically with a certain energy shift ΔV on both half periods. A numerical analysis with finite $l\neq 0$ reveals a deterioration of the main motor characteristics in accordance with the second criterion for the maximum efficiency [12]. The sawtooth potential involved here has no additional barrier V_0 and hence the first condition cannot be satisfied from the formal point of view. At the same time, the parameter ΔV of an extremely asymmetric sawtooth potential plays a twofold part: as the value of ΔV increases, the system moves away from equilibrium (where the flux vanishes) and a potential barrier arises which blocks the backward flux. On the one hand, efficiency is known to be reduced on moving away from equilibrium [10,11]; on the other hand, it grows with the degree of blocking of the backward flux. It is the competition of the two trends that governs the efficiency so that the resulting value of η_m slowly approaches unity roughly following the law $1-\ln(2\beta\Delta V)/\beta\Delta V$ [in accordance with Eq. (31)].

The question arises of whether such behavior of the efficiency is unique for an extremely asymmetric sawtooth potential or is typical of any potentials that are shaped identically with a certain energy shift on both half periods. For better understanding, it is expedient to invoke the kinetic description of a flashing ratchet with periodic two-well potentials as in Fig. 1(a) [11]. This approach is advantageous because the main trends in the quantities concerned are most simply expressible in terms of two main parameters of the potential profile, viz., the difference of the barrier heights, $V-v$, and the difference of the well minima, $2u$. The kinetic description holds good when the potential switching frequency γ is small as compared to the intrawell relaxation frequency k_0. However, if the lower barriers v are also high enough, the dimensionless parameter $(k_0/\gamma)\exp(-v)$ [with $\beta=(k_BT)^{-1}\equiv 1$] can be neglected and the previously published formula for efficiency [see Eq. (21) in Ref. [11]] is rewritten as

$$\eta(f)=\frac{f}{u}\frac{\sinh(u-f)-e^{-V+v}\sinh(u+f)}{\sinh(u-f)+e^{-V+v}\sinh(u+f)}, \qquad (32)$$

where $f=FL/2$.

Figure 7 presents the maximum η_m of the function $\eta(f)$ plotted versus $2u$ in two cases. The first implies a constant difference $V-v$. As the parameter u rises, the system moves away from equilibrium and the value of η_m decreases as

$$\eta_m\approx 1-4\exp[-(V-v)/2](1+u^2/3),\quad V-v\gg 1,\quad u\ll 1. \qquad (33)$$

Note that both half periods of the asymmetric potential profile include, with an energy shift of $2u$, the same portions near the minima of the potential well. Another attribute of the potential is a high barrier V blocking the backward flux. In the second case we assume that the potential profile $U^+(x)$ on the interval $(L,2L)$ is the same as on the interval $(0,L)$ but displaced downward by the energy $2u$ [see Fig. 1(a)]. Hence it follows in terms of barrier heights and well depths that $V-v=2u$. It can be shown that this condition significantly changes the f dependence of the power expended for the potential switches: the function $W_{in}(f)$ becomes negative at $V-v>2u$ for sufficiently large $f>u$, whereas it is always positive at $V-v<2u$ [18]. The latter case most closely resembles an extremely asymmetric sawtooth potential, and it is of interest to clarify whether this resemblance extends to the behavior of efficiencies.

At $u\ll 1$ we have $\eta_m\approx u^2/4$, which is at variance with the trend specified by Eq. (27) for an extremely asymmetric sawtooth potential; the distinction arises from the possibility of surmounting a thermally activated barrier in the two-well potential. However, the opposite limit, $u\gg 1$, provides

$$\eta_m \approx 1 - \frac{1 + \ln 4u}{2u}, \quad u - f_m \approx \frac{1}{2}\ln 4u, \tag{34}$$

in complete accord with the asymptotic (31) [in view of the correspondence between the parameters of the two models $\Delta V = 2u$ and $\varphi_m = 2(u - f_m)$].

Thus, we arrive at the general inference that two approaches can be invoked to design a high-efficiency flashing ratchet with periodic potentials fluctuating via random half-period shifts. The first approach can be realized in the near-equilibrium region and implies introduction of a high barrier V_0 blocking the reverse motion of a Brownian particle. It is also required that both half periods of the asymmetric potential profile include the same flat portions mutually shifted by a certain energy. Another route to high efficiency is possible far from equilibrium and typical of strongly asymmetric potentials which are shaped identically on both half periods with a large energetic shift ΔV. The two mechanisms considered exhibit radically different limiting behavior of the maximum possible efficiency: $\eta_m \sim 1 - \exp(-\beta V_0/2)$ for the former and $\eta_m \sim 1 - \ln(2\beta\Delta V)/\beta\Delta V$ for the latter.

ACKNOWLEDGMENTS

The authors thank Dr. M. L. Dekhtyar for her helpful comments on the manuscript. This work was supported by Academia Sinica. V.M.R. gratefully acknowledges the kind hospitality received from the Institute of Atomic and Molecular Sciences.

[1] R. D. Astumian and M. Bier, Phys. Rev. Lett. **72**, 1766 (1994).
[2] J. Prost, J.-F. Chauwin, L. Peliti, and A. Ajdari, Phys. Rev. Lett. **72**, 2652 (1994).
[3] J.-F. Chauwin, A. Ajdari, and J. Prost, Europhys. Lett. **27**, 421 (1994).
[4] R. D. Astumian, Science **276**, 917 (1997).
[5] F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).
[6] J. M. R. Parrondo and B. J. de Cisneros, Appl. Phys. A: Mater. Sci. Process. **A75**, 179 (2002).
[7] P. Reimann, Phys. Rep. **361**, 57 (2002).
[8] T. Y. Tsong and R. D. Astumian, Prog. Biophys. Mol. Biol. **50**, 1 (1987).
[9] R. D. Astumian, J. Phys. Chem. **100**, 19075 (1996).
[10] A. Parmeggiani, F. Jülicher, A. Ajdari, and J. Prost, Phys. Rev. E **60**, 2127 (1999).
[11] Yu. A. Makhnovskii, V. M. Rozenbaum, D.-Y. Yang, S. H. Lin, and T. Y. Tsong, Phys. Rev. E **69**, 021102 (2004).
[12] V. M. Rozenbaum, Pis'ma Zh. Eksp. Teor. Fiz. **79**, 475 (2004) [JETP Lett. **79**, 388 (2004)].
[13] T. Y. Tsong and R. D. Astumian, Bioelectrochem. Bioenerg. **15**, 457 (1986).
[14] V. S. Markin, T. Y. Tsong, R. D. Astumian, and R. J. Robertson, Chem. Phys. **93**, 5062 (1990).
[15] V. S. Markin and T. Y. Tsong, Biophys. J. **59**, 1308 (1991).
[16] V. S. Markin and T. Y. Tsong, Bioelectrochem. Bioenerg. **26**, 251 (1991).
[17] Y. Chen and T. Y. Tsong, Biophys. J. **66**, 2151 (1994).
[18] V. M. Rozenbaum, D.-Y. Yang, S. H. Lin, and T. Y. Tsong, J. Phys. Chem. B **108**, 15880 (2004).
[19] J. M. R. Parrondo, J. M. Blanko, F. J. Chao, and R. Brito, Europhys. Lett. **43**, 248 (1998).
[20] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).